Chapter I

1. First order Linear equation

$y' + p(x)y = q(x)$

general sol., $y = e^{-\int p(x)}\left(C + \int e^{\int p(t)} q(t)\,dt\right)$

~~Becaf~~ ~~ke~~ careful. $\underline{y' = p(x)y + q(x)}$

2. separable eq

→ consider y as variable eg, $\frac{dy}{dx} = \frac{y+1}{y+2}$

$$\underline{\frac{dx}{dy} = \frac{1}{y+1} + 1}$$

3. Homogeneous eq. if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

Let $u = y/x$, $u(x)$ $y = ux$ $\frac{dy}{dx} = x\frac{du}{dx} + u = f(u)$

$\Rightarrow \frac{du}{dx} = \frac{f(u) - u}{x}$ → separable eq

4 ~~Not~~ Exact eq → $\underline{M\,dx + N\,dy = 0}$

$M_y = N_x$ When an eq is not exact, find an integrating factor

if μ ~~is~~ depends only on x. ← $\frac{M_y - N_x}{N}$ depends only on x

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \leftarrow \text{separable eq}$$

μ — — y ← $\frac{N_x - M_y}{M}$ depen on ~~x~~ y

eg, $(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$

Exercise:

$$\cancel{\left(8x+\frac{6}{y}\right)dx+\left(\frac{x^2}{y}+\frac{3y}{x}\right)\frac{dy}{dx}=0}$$

$$\cancel{M_y=-\frac{6}{y^2}\quad N_x=\frac{2x}{y}-\frac{3y}{x^2}\neq M_y}$$

$$\cancel{\frac{M_y-N_x}{N}=}$$

$$\left[\frac{4x^3}{y^2}+\frac{3}{y}\right]dx+\left[\frac{3x}{y^2}+4y\right]dy=0$$

$M_y=-8\dfrac{x^3}{y^3}-\dfrac{3}{y^2}\quad N_x=\dfrac{3}{y^2}$ ← Not exact ; give results

$$\frac{M_y-N_x}{N}=\frac{-\frac{8x^3}{y^3}-\frac{6}{y^2}}{\frac{3x}{y^2}+4y}\quad X$$

$$\frac{N_x-M_y}{M}=\frac{\frac{6}{y^2}+\frac{8x^3}{y^3}}{\frac{4x^3}{y^2}+\frac{3}{y}}=2\cdot\frac{1}{y}$$

choose $\mu=y^2$ $\quad[4x^3+3y]dx+(3x+4y^3)dy=0$ ← Exact

$$\boxed{x^4+3xy+y^4=C}$$

High order Liner eq

1. Structure of sols

n-th Homo eq $\Rightarrow$ n - L.I sols.

n-th InHomogeneous eq $\longrightarrow$

general sol = general sol of Homo + particular.

2. Solve:

For Homogeneous:

① Constant coefficients $\Leftarrow$ characteristic eq

② Non-constant $\Leftarrow$ Reduction of order / Abel's formula

For InHomogeneous

① Undetermined coefficients ☆

② Variation of parameters. ☆

Exercise: Suppose y_1, y_2 are sols of $y'' + p(t)y' + q(t)y = g(t)$

Then ① $y_1 + y_2$ is sol of InHomogeneous

② $y_1 - y_2$ is sol of InHomog

③ $y_1 + y_2$ is sol of Homo

④ $y_1 - y_2$ is sol of Homo

pay attention: ~~the~~ use Abel's formula and variation of parameters.

be careful for that whether the coefficients of n-th derivative is 1 !

Eg. $4y''-4y'+y=8e^{\frac{t}{2}}$

$W=C\cdot e^{\int -1 dx}$

$y=\sum y_i\int\frac{W_i g}{W}ds$ $\quad g=2e^{\frac{t}{2}}$

Step ① Find a particular of homo eq
② Reduction of order or Abel's for fundamental set.
③ Use variation of parameter.

For undetermined coefficients.

Eg: determine the form of particular set of following eq:

① $x''-2x'+2x=e^{t}+\sin t+e^{t}(\sin t+\cos t)$

$Y(t)=Ae^{t}+[B\sin t+B'\cos t]+te^{t}[\sin t\cdot C+\cos t\cdot D]+[Ee^{t}\cos 2t+E'e^{t}\sin 2t]$

② $x^{(3)}-x''-x'+x=e^{t}\sin t+te^{t}+te^{-t}+e^{-t}\sin t$

$Y(t)=[Ae^{t}\sin t+A'e^{t}\cos t]+t^2(Bt+C)e^{t}+t(Dt+E)e^{-t}+[Fe^{-t}\sin t+F'e^{-t}\cos t]$

③ $x^{(3)}-3x''+3x'-x=e^{t}(t+8)+t^2+\sin t$

$Y(t)=t^3e^{t}(At+B)+(Ct^2+Dt+E)+[F\sin t+G\cos t]$

For variation of parameter $\quad g=3-\frac{1}{t^2}$

$t^2y''-2y=3t^2-1\quad t>0,\quad y_1=Y_1(t)=t^2\qquad t^2y''-2y=0$

$y=v\varphi_1\quad y''=\varphi_1 v''+2\varphi_1'v'+\varphi_1''v$

Choose $y_2=\frac{1}{t}\quad W_1=\begin{vmatrix}0&\frac{1}{t}\\1&-\frac{1}{t^2}\end{vmatrix}=-\frac{1}{t}$

$t^2(\varphi_1v''+2\varphi_1'v'+\varphi_1''v)-2v\varphi_1=0$

$W_2=t^2$

$\Rightarrow t^2\varphi_1v''+2\varphi_1'v'=0\qquad t^2v''+4tv'=0$

$W(y_1,y_2)=\begin{vmatrix}t^2&\frac{1}{t}\\2t&-\frac{1}{t^2}\end{vmatrix}=-3$

$Y^*=\sum y_m\int\frac{g(s)W_m(s)}{W(s)}ds$
$=t^2\int(3-\frac{1}{s^2})(-\frac{1}{s})/_{-3}ds$

$w=v'\quad w'=-\frac{4}{t}w\quad$ choose $w=\frac{c}{t^4}\quad v=\frac{c}{3}-\frac{1}{3t^3}+C_2$

$y_2=-\frac{C}{3}\frac{1}{t}+Ct^2$

$$Y^* = \sum y_n \int \frac{g(s)\, W_n(s)}{W(s)}\, ds$$

$$= t^2 \int \left(3 - \frac{1}{t^2}\right)\left(-\frac{1}{t}\right) \Big/ {-3}\, dt + \frac{1}{t} \int \frac{\left(3 - \frac{1}{t^2}\right) t^2}{-3}\, ds$$

$$= \frac{t^2}{3} \int \left(\frac{3}{t} - \frac{1}{t^3}\right) dt + \frac{1}{3t} \int (1 - 3t^2)\, dt$$

$$= \frac{t^2}{3}\left(3 \ln|t| + \frac{1}{2}\frac{1}{t^2}\right) + \frac{1}{3t}\left(t - t^3\right) + c$$

$$= t^2 \ln|t| + \frac{1}{6} + \frac{1}{3} - \frac{1}{3}t^2$$

$$= t^2 \ln|t| + \frac{1}{2} - \frac{1}{3}t^2 \rightarrow \text{multiply of } y_1$$

Laplace Transform Solve O.D.E with Step Forcing Functions (take constant-coefficients eq for eg).

$$a y'' + b y' + c y = g(t) \quad y(0) = A, \; y'(0) = B$$

Step. ① write $g(t)$ as multiply of $u_c(t)$

as $g(t) = [u_{c_1}(t) f_1(t - c_1) - u_{c_2}(t) f_2(t - c_2)] \rightarrow \mathcal{L}\{g\} = [e^{-c_1 s} - e^{-c_2 s}] \cdot \mathcal{L}\{f_1\}\ \mathcal{L}\{f_2\}$

② find $Y(t)$ in terms of $\mathcal{L}\{g\}$. $\mathcal{L}\{h(t)\}$ $y(t) = u_{c_1}(t) h(t - c_1) - u_{c_2}(t) h(t - c_2)$

$$Y(t) = \frac{\mathcal{L}\{g\} + A + B}{Ps^2 + Qs + R} = [e^{-c_1 s} - e^{-c_2 s}] \cdot H(s)$$

L-Transf...

③ ~~The~~ Find inverse L-Transf of $Y(t)$ [use table]

Solve the L-T (familiar)

~~Chapter 3~~

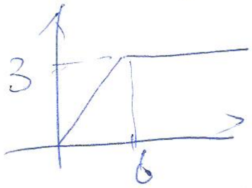
~~1. Second Order Linear eq~~

~~Structure of sol~~

For L-Trans of ble.
should be familar.

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$

Example. $y'' + y = g(t) \quad y'(0) = 1 \quad y(0) = 0 \quad g(t) = \begin{cases} \frac{t}{2} & 0 \le t \le 6 \\ 3 & t \ge 6 \end{cases}$



① $g(t) = \frac{t}{2} - u_6(t)(t-6)/2 = \frac{1}{2}[t - u_6(t)(t-6)]$

② $\mathcal{L}\{y''\} + \mathcal{L}y = \mathcal{L}[g(t)]$

$\Rightarrow s^2 \mathcal{L}[y] + \mathcal{L}[y] - s y(0) - y'(0) = \frac{1}{2}\left[\frac{1}{s^2} - \frac{e^{-6s}}{s^2}\right]$

$(s^2+1)\mathcal{L}[y] = \frac{1 - e^{-6s}}{2s^2} + 1$

$\mathcal{L}[y] = \frac{1}{s^2+1} + \frac{1 - e^{-6s}}{2(s^2+1)s^2} = \frac{1}{s^2+1} + [1 - e^{-6s}]\, H(s)$

③

$\underline{\sin t} \xleftarrow{\mathcal{L}^{-1}}$

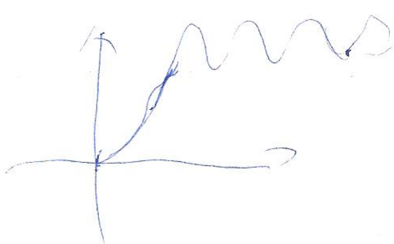
$H(s) = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$

$\mathcal{L}^{-1} \Downarrow$

$\frac{1}{2}[t - \sin t]$

Then $\underline{y = \sin t + \frac{1}{2}[t - \sin t] - \frac{1}{2} u_6(t)[(t-6) - \sin(t-6)]}$ ✓

$= \frac{1}{2}[t + \sin t] - \frac{1}{2} u_6(t)[(t-6) - \sin(t-6)]$



System of First Order

I) $x' = Ax$ (const coeff again)

A: eigenvalues

(1) If A has n distinct eigenvalues

$r_1 \cdots r_n$, $\xi_1 \cdots \xi_n$ $\longrightarrow$ $x = c_1 \xi_1 e^{r_1 t} + \cdots c_n \xi_n e^{r_n t}$

(2) if A has repeated eigenvalues.

(1) A can be diagonalized

$r_i \to n_i$, $\xi^{(i)}_{n_1} \cdots \xi^{(i)}_{n_i}$

For r_i $\cancel{x^{(i)} = e^{r_i t}(c_{n_1} \xi^{(i)}_{n_1} + c_{n_2} \xi)}$

$x^{(i)}_{n_1} = \xi^{(i)}_{n_1} e^{r_i t}$ $x^{(i)}_{n_2} = \xi^{(i)}_{n_2} e^{r_i t} \cdots$

2. A can't be diagonalized.

Three methods.

Careful, (1) The order of generalized eigenvector

e.g., $A \sim \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$\cancel{(A-I)^3 \xi}$ $\ker(A-I)^3 \supset \ker(A-I)^2 \supset \cdots \supset \ker(A-I)$

ξ. $(A-I)\xi$, $(A-I)^2\xi$

$T = ((A-I)^2\xi, (A-I)\xi, \xi)$ $T^{-1}AT = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$T = (\xi, (A-I)\xi, (A-I)^2\xi)$ $T^{-1}AT = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$

(2) $\xi \in \ker(A - \lambda I)$ $\quad \xi, \eta, \beta \Rightarrow \begin{cases} (A-\lambda I)\beta = \eta \\ (A-\lambda I)\eta = \xi \end{cases}$

$(A-\lambda I)\eta = \xi$

$(A-\lambda I)\beta = \eta$ $\Rightarrow T = (\beta, \eta, \xi)$ $\times$ $T = (\xi, \eta, \beta)$

The limit function is

$\phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k+1)!} \frac{t^{k+1}}{2^{k+1}}$

$= 4 \cdot \sum_{k=2}^{n} \frac{(-1)^k}{k! \, 2^k} \cdot$

$= 4\left(e^{-\frac{t}{2}} - 1 + \frac{t}{2}\right)$

$= 4e^{-\frac{t}{2}} + 2t - 4$

Chap. –

$|\phi(t) - \psi(t)|$

$\leq \int_0^t \left|\frac{\phi(s)}{2} - \frac{\psi(s)}{2}\right| ds$

$\leq \frac{1}{2} \int_0^t |\phi(s) - \psi(s)| ds$

Let $U = \int_0^t |\phi(s) - \psi(s)| ds$

Exercise

Homework 4 pro 1, 2

Existence and Uniqueness theorem.

Use picard's iteration method to show the existence and uniqueness of sol.

$t^* = t - a$

Eg. for initial $y(a) = b \Rightarrow$ we transform $y^* = y - b$

to change the ini cd. to be $y(0)=0$

$y' = t^2 + y^2$. $y(0)=0$

$y = \phi(t) = \int_0^t s^2 + \phi^2(s)\, ds$

$\phi_0(s) = 0 \quad \phi_1(t) = \int_0^t s^2 + 0\, ds = \frac{1}{3}t^3$

$\phi_2(t) = \int_0^t s^2 + \left(\frac{1}{3}s^3\right)^2 ds = \frac{1}{3}t^3 + \frac{1}{3^2}\cdot\frac{1}{7}t^7$

~~guess. $\phi_n = 2\left(\frac{t^3}{3!} + \frac{t^4}{4!} + \dots + \frac{t^n}{n!} + \frac{t^{n+2}}{(n+2)!}\right)$~~

~~Use mathematical induction to verify your guess~~

~~$\phi_{n+1} = \int_0^t s^2 + \phi_n^2$~~

$\phi_3(t) = \int_0^t s^2 + \left(\frac{1}{3}s^3 + \frac{1}{3^2}\cdot\frac{1}{7}s^7\right)^2 ds = \int_0^t \frac{1}{3}t^3 + \left(\frac{1}{3}s^3\right)^2 + 2\cdot\frac{1}{3}\cdot\frac{1}{3^2}\cdot\frac{1}{7}s^{10} + \frac{1}{3^4}\cdot\frac{1}{7^2}s^{14}$

guess

$\phi_n(t) = \sum_{k=1}^{n} \frac{t^{k+1}}{(k+1)!}(-1)^{k+1}\frac{1}{2^{k-1}}$

verify.

$\phi_{n+1}(t) = \int_0^t -\frac{1}{2}\phi_n(s) + s$

$= \int_0^t \sum_{k=1}^{n} (-1)^{k+2}\frac{1}{2^k}\frac{s^{k+1}}{(k+1)!} + s\, ds$

$= \int_0^t \frac{t^2}{2} + \sum_{k=1}^{n} \frac{s^{k+2}}{(k+2)!}\frac{(-1)^{k+2}}{2^k}$

$= \dots$

$y' = -\frac{y}{2} + t \quad y(0)=0$

$\phi(t) = \int_0^t -\frac{\phi(s)}{2} + s\, ds \quad \phi_0(t) = 0 \quad \phi_1(t) = \int_0^t s\, ds = \frac{t^2}{2}$

$\phi_2(t) = \int_0^t -\frac{\phi_1(s)}{2} + s\, ds = \int_0^t -\frac{s^2}{4} + s\, ds = \frac{t^2}{2} - \frac{1}{3\cdot 4}t^3$

$\phi_3(t) = \int_0^t -\frac{\phi_2(s)}{2} + s\, ds = \int_0^t -\frac{s^2}{4} + \frac{1}{2\cdot 3\cdot 4}s^3 + s\, ds = \frac{t^2}{2} - \frac{1}{3\cdot 4}t^3 + \frac{1}{2\cdot 3\cdot 4\cdot 4}t^4$

$\phi_4(t) = \int_0^t -\frac{\phi_3(s)}{2} + s\, ds = \frac{t^2}{2} - \frac{1}{3\cdot 4}t^3 + \frac{1}{2\cdot 3\cdot 4\cdot 4}t^4 - \frac{t^5}{2\cdot 3\cdot 4\cdot 5\cdot 2^3} = \frac{t^2}{2} - \frac{1}{2\cdot 3}\cdot\frac{t^3}{2} + \frac{1}{2\cdot 3\cdot 4}\cdot\frac{t^4}{2^2} - \dots$

ratio test

$\left|\frac{\frac{t^{k+2}}{(k+2)!}\frac{(-1)^{k+2}}{2^k}}{\frac{t^{k+1}}{(k+1)!}\frac{(-1)^{k+1}}{2^{k-1}}}\right| = \frac{t}{2(k+2)} \to 0$ converge